The Tertiary Mirror Equations of Motion for an “off-altitude-axis” Nasmyth Focus

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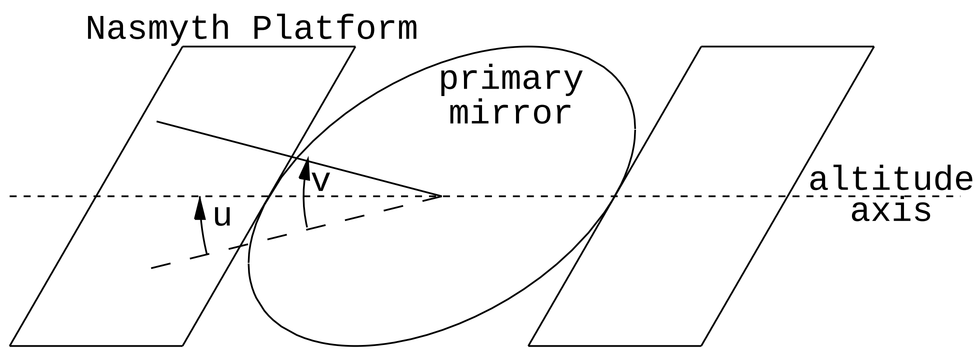
1 Introduction

Large alt-azimuth mounted telescopes can take advantage of the Nasmyth focus for additional instruments. In the case of the Keck telescopes these instruments are mounted to the Nasmyth platforms, which move with the telescope in azimuth but remain stationary when the telescope's altitude is changed. Generally the instruments will be aligned with the altitude axis of the telescope, in which case the tertiary mirror does not need to be moved independently of the primary mirror. The size and mass of instruments scale approximately with the size of the telescope. For the proposed *California Extremely Large Telescope* (C.E.L.T.), with its 30m diameter primary mirror, the projected size and mass would make a swapping of the instruments difficult and time consuming. For this reason it has been proposed to allow for additional positions on the Nasmyth platforms, which would necessarily lie away from the altitude axis. By changing the configuration of the tertiary mirror different instruments could be accessed without having to take the time to interchange their positions.



An "off-altitude-axis" position on the Nasmyth platforms brings with it three complications:

1. The position of the tertiary mirror is dependent on altitude and must be adjusted as the telescope tracks an object.

2. For certain positions on the Nasmyth platform the tertiary mirror has to be moved in such a way that some vignetting occurs.

3. The amount of field rotation will vary for different Nasmyth positions.

In this paper I will address all three of these issues and describe the mathematical relations, independently of the actual design of the telescope.

2 Background

The azimuth (A) and altitude (a) of the primary mirror and the amount of field rotation (p) depend on the latitude (ϕ) of the telescope, and the declination (δ) and hour angle (h) of the target. The dependences of A, a, p, and their first and second time derivatives are calculated in Nelson (1981) and are given by equations 1-9. Here the natural sidereal rate is given by $\dot{h} = \omega_0 \simeq 15^\circ h^{-1}$.

$$a = \sin^{-1}\left[\sin\delta\sin\phi + \cos\delta\cos\phi\cos h\right] \tag{1}$$

$$A = -\tan^{-1}\left[\frac{\cos\delta\sin h}{\sin\delta\cos\phi - \cos\delta\sin\phi\cos h}\right] \tag{2}$$

$$p = \tan^{-1}\left[\frac{\sin h}{\tan\phi\cos\delta - \sin\delta\cos h}\right] \tag{3}$$

$$\frac{\dot{a}}{\omega_0} = \sin A\cos\phi \tag{4}$$

$$\frac{\dot{A}}{\omega_0} = \sin\phi - \tan a\cos A\cos\phi \tag{5}$$

$$\frac{\dot{p}}{\omega_0} = -\frac{\cos\phi\cos A}{\cos a} \tag{6}$$

$$\frac{\ddot{a}}{\omega_0^2} = \frac{\dot{A}}{\omega_0}\cos A\cos\phi \tag{7}$$

$$\frac{\ddot{A}}{\omega_0^2} = -\tan^2 a\sin 2A\cos^2\phi + \frac{1}{2}\tan a\sin A\sin 2\phi - \frac{1}{2}\sin 2A\cos^2\phi \tag{8}$$

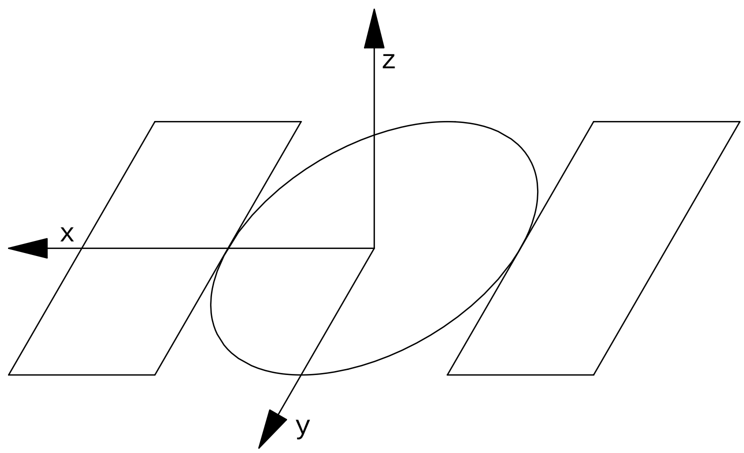
$$\frac{\ddot{p}}{\omega_0^2} = \frac{\sin A\sin 2\phi}{2\cos a} - \frac{\sin a\sin 2A\cos^2\phi}{\cos^2 a} \tag{9}$$

Note that $\dot{A}$, $\dot{p}$, $\ddot{a}$, $\ddot{A}$, and $\ddot{p}$ experience singularities as $a \to \frac{\pi}{2}$. Due to physical limitations of the telescope drives each of these terms will generate its own blind spot. As is shown in later sections the necessary movement of the tertiary mirror is not subject to any additional blind spots.

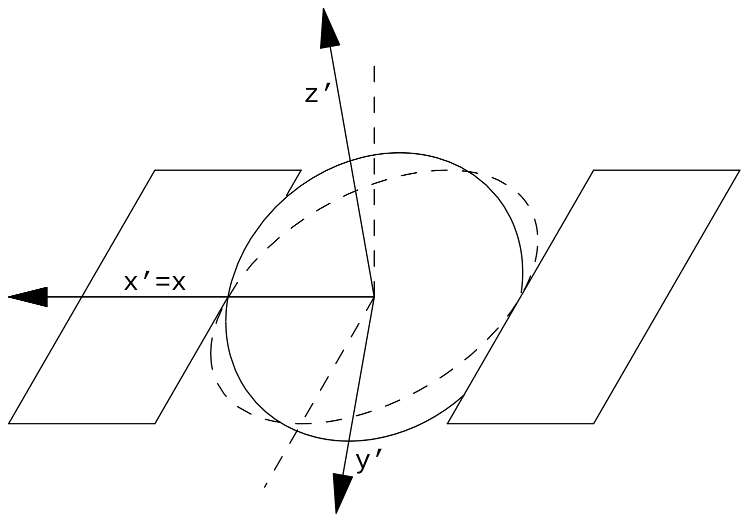
3 Coordinate Systems

In the following calculations I will make use of three distinct coordinate systems. All three have a coincident origin, but are "tied" to different elements of the telescope:

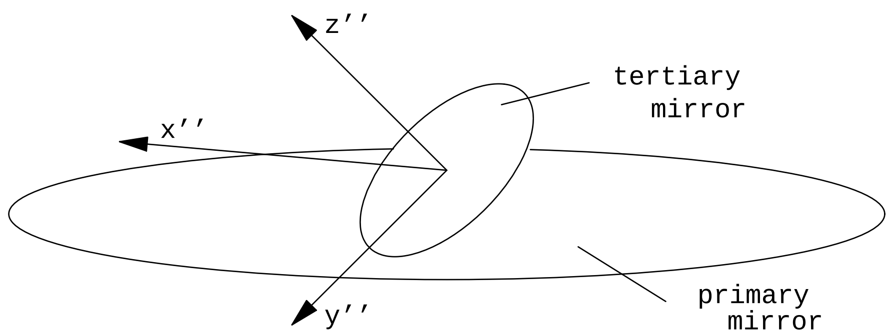
1. Unprimed coordinates (x, y, z) are fixed with respect to the Nasmyth platforms.



2. Single primed coordinates (x', y', z') are fixed with respect to the primary mirror.



3. Double primed coordinates (x'', y'', z'') are fixed with respect to the tertiary mirror.



In these coordinates the altitude axis is identical to both the x and x' axes, and the optical axis is the z' axis. Since the Nasmyth platforms move with the primary mirror in azimuth, the primed coordinate system is related to the unprimed by only a rotation around the altitude axis by an angle Δa. When the primary mirror is pointed at the zenith ($a = \frac{\pi}{2}$) the primed and unprimed coordinate systems are identical. The double primed coordinate system, however, is related to the primed one by two rotations: one around the optical axis (z') by an angle λ and the other around the x'' axis by an angle μ. It is one of the purposes of this paper to describe the equations of motion of the tertiary mirror, i.e. the dependence of λ, μ, and their time derivatives on a, u, and v.

4 Tertiary Mirror Equations of Motion ($v = 0$)

For simplicity I will consider in this section the case where the additional instrument positions still lie in the orgininal plane ($v = 0$). In section 5 I will then present the results for the general case.
The initial position of the tertiary mirror ($\lambda, \mu = 0$) is defined such that it will reflect the incoming lightray onto the positive x-axis ($u = 0$). The orientation of the tertiary mirror is fixed by the direction of the normal (n_3). The initial normal is given by:

$$\hat{n}_3 = n_3(a, u = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \cos a \\ \sin a \end{pmatrix} \tag{10}$$

In order for the incoming lightray to be reflected onto an arbitrary position on the Nasmyth platform the tertiary mirror must have a position specified by this normal:

$$n'_3 = n_3(a, u) = \frac{1}{\sqrt{2(1 + \cos a \sin u)}} \begin{pmatrix} \cos u \\ \cos a + \sin u \\ \sin a \end{pmatrix} \tag{11}$$

All that needs to be done now is to find λ and μ, such that $\hat{n}_3$ is rotated onto n'_3. I will employ the following strategy to solve this problem:

- Determine the matrix representing the rotation of a vector around the optical axis in unprimed coordinates, call it $R_{z'}$.

- Determine the matrix representing the rotation of a vector around the x'' axis in unprimed coordinates, call it $R_{x''}$.

- Solve $R_{x''} \cdot R_{x'} \cdot \hat{n}_3 = n'_3$ for $\lambda(a, u)$ and $\mu(a, u)$.

4.1 Rotation about z' axis

The matrix representing the rotation of a vector around the z' axis is obtained by the following three steps:

i) transform into primed coordinate system, i.e. $(0, \cos a, \sin a) \rightarrow (0, 0, 1)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin a & -\cos a \\ 0 & \cos a & \sin a \end{pmatrix} \begin{pmatrix} 0 \\ \cos a \\ \sin a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

ii) rotate about z-axis by λ degrees (still in primed coordinate system).

$$\begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

iii) transform back into unprimed coordinate system, i.e. $(1, 0, 0) \rightarrow (0, \cos a, \sin a)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin a & \cos a \\ 0 & -\cos a & \sin a \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos a \\ \sin a \end{pmatrix} \tag{14}$$

Thus a rotation about the z' axis by an angle λ is represented by this matrix in unprimed coordinates:

$$R_{z'} = \begin{pmatrix} \cos\lambda & -\sin a \sin\lambda & \cos a \sin\lambda \\ \sin a \sin\lambda & \sin^2 a \cos\lambda + \cos^2 a & \sin a \cos a (1 - \cos\lambda) \\ -\cos a \sin\lambda & \sin a \cos a (1 - \cos\lambda) & \cos^2 a \cos\lambda + \sin^2 a \end{pmatrix} \tag{15}$$

4.2 Rotation about x'' axis

First we need to determine the representation of x'' in the unprimed coordinate system. As mentioned above the double primed coordinate system is tied to the tertiary mirror and hence depends on both λ and μ. But since the x'' axis is the rotation axis for the angle μ, it will depend only on λ. In primed coordinates x'' is given by $(\sin\lambda, -\cos\lambda, 0)$, since when $\lambda = 0$ the tertiary mirror x-axis coincides with the negative primary mirror y-axis ($x'' = -y'$). x'' is transformed to the unprimed coordinate system by the matrix in eq.14. Thus x'' in unprimed coordinates is given by $(\sin\lambda, -\sin a \cos\lambda, \cos a \cos\lambda)$.

Now we are prepared to determine the matrix representing a rotation around the x'' axis in unprimed coordinates. In analogy to the procedure employed in 4.1 this is achieved by three steps:

i) transform to a frame where $(\sin\lambda, -\sin a\cos\lambda, \cos a\cos\lambda)$ is $(1,0,0)$. For simplicity I will pick the double primed coordinate system with $\mu = 0$.

$$\begin{pmatrix} \sin\lambda & -\cos\lambda\sin a & \cos a\cos\lambda \\ \frac{1}{\sqrt{2}}\cos\lambda & \frac{-1}{\sqrt{2}}(\cos a - \sin a\sin\lambda) & \frac{-1}{\sqrt{2}}(\sin a + \cos a\sin\lambda) \\ \frac{1}{\sqrt{2}}\cos\lambda & \frac{1}{\sqrt{2}}(\cos a + \sin a\sin\lambda) & \frac{1}{\sqrt{2}}(\sin a - \cos a\sin\lambda) \end{pmatrix} \tag{16}$$

ii) rotate about x-axis by μ degrees (still in double primed coordinate system).

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu & \cos\mu \end{pmatrix} \tag{17}$$

iii) transform back into unprimed coordinate system.

$$\begin{pmatrix} \sin\lambda & \frac{1}{\sqrt{2}}\cos\lambda & \frac{1}{\sqrt{2}}\cos\lambda \\ -\cos\lambda\sin a & \frac{-1}{\sqrt{2}}(\cos a - \sin a\sin\lambda) & \frac{1}{\sqrt{2}}(\cos a + \sin a\sin\lambda) \\ \cos a\cos\lambda & \frac{-1}{\sqrt{2}}(\sin a + \cos a\sin\lambda) & \frac{1}{\sqrt{2}}(\sin a - \cos a\sin\lambda) \end{pmatrix} \tag{18}$$

Thus a rotation about the x'' axis by an angle μ is represented by this rather complicated matrix in unprimed coordinates:

$$R_{x''} = \begin{pmatrix} \cos^2\lambda\cos\mu + \sin^2\lambda & \cos\lambda(\sin a\sin\lambda(\cos\mu - 1) - \cos a\sin\mu) & -\cos\lambda(\sin\lambda\cos a(\cos\mu - 1) + \sin a\sin\mu) \\ \cos\lambda(\sin a\sin\lambda(\cos\mu - 1) + \cos a\sin\mu) & \cos^2 a\cos\mu + \sin^2 a(\cos^2\lambda + \sin^2\lambda\cos\mu) & -\cos^2\lambda\sin 2a\sin^2\frac{\mu}{2} - \sin\lambda\sin\mu \\ -\cos\lambda(\sin\lambda\cos a(\cos\mu - 1) - \sin a\sin\mu) & -\cos^2\lambda\sin 2a\sin^2\frac{\mu}{2} + \sin\lambda\sin\mu & \cos\mu\sin^2 a + \cos^2 a(\cos^2\lambda + \sin^2\lambda\cos\mu) \end{pmatrix} \tag{19}$$

4.3 Equations of Motion

We are now prepared to apply the two rotations $(R_{z'}, R_{x''})$ to the initial normal of the tertiary mirror $(\hat{n}_3)$, equate the resulting vector to the final tertiary normal (n'_3), and solve for λ and μ.

$$R_{x''} \cdot R_{z'} \cdot \hat{n}_3 = n'_3 \tag{20}$$

This leads to the following three equations:

$$\text{I}\ \cos\lambda(\cos\mu - \sin\mu) = \frac{\cos u}{\sqrt{1+\cos a \sin u}}$$

$$\text{II}\ \sin a \sin\lambda(\cos\mu - \sin\mu) + \cos a(\cos\mu + \sin\mu) = \frac{\cos a + \sin u}{\sqrt{1+\cos a \sin u}}$$

$$\text{III}\ -\cos a \sin\lambda(\cos\mu - \sin\mu) + \sin a(\cos\mu + \sin\mu) = \frac{\sin a}{\sqrt{1+\cos a \sin u}}$$

Solving these three equations for λ and μ, and then taking time derivatives I arrive at the following solutions:

$$\sin\lambda = \frac{\sin a \tan u}{\sqrt{1+\sin^2 a \tan^2 u}}$$

$$\cos\lambda = \frac{1}{\sqrt{1+\sin^2 a \tan^2 u}} \tag{21}$$

$$\dot\lambda = \dot a \frac{\cos a \tan u}{1+\sin^2 a \tan^2 u} \tag{22}$$

$$\ddot\lambda = \frac{\ddot a}{\dot a}\dot\lambda - \dot\lambda^2 \sin a \tan u \left[1 + \frac{1}{\sin^2 u \cos^2 a}\right] \tag{23}$$

$$\sin\mu = \frac{1}{2}\left[\sqrt{1+\cos a \sin u} - \sqrt{1-\cos a \sin u}\right]$$

$$\cos\mu = \frac{1}{2}\left[\sqrt{1+\cos a \sin u} + \sqrt{1-\cos a \sin u}\right] \tag{24}$$

$$\dot\mu = -\frac{\dot a \sin a \sin u}{2\sqrt{1-\cos^2 a \sin^2 u}} \tag{25}$$

$$\ddot\mu = \frac{\ddot a}{\dot a}\dot\mu + \dot\mu \dot a \cot a \left[1 - 4\left(\frac{\dot\mu}{\dot a}\right)^2\right] \tag{26}$$

One can immediately see that no additional blindspots are introduced, since none of the expressions have singularities for $u \neq \frac{\pi}{2}$. The reader is encouraged to check these results for two limiting cases. With the primary mirror pointing at zenith ($a = \frac{\pi}{2}$) a rotation about the x'' axis is unnecessary ($\mu = 0$) and the rotation around the optical axis is all that's needed ($\lambda = u$). On the other hand, when the primary is pointing at the horizon ($a = 0$), the rotation about the optical axis is unnecessary ($\lambda = 0$) and the tertiary needs to be rotated about the x'' axis by $\mu = \frac{u}{2}$.

5 General Tertiary Mirror E.O.M. $(u, v \neq 0)$

When allowing for additional instrument positions that are elevated with respect to the traditional "on-altitude-axis" position, the equations get a little more complicated. The normal fixing the orientation of the tertiary mirror is now:

$$n_3' = \frac{1}{\sqrt{2(1 + \cos a \sin u \cos v + \sin a \sin v)}} \begin{pmatrix} \cos u \cos v \\ \cos a + \sin u \cos v \\ \sin a + \sin v \end{pmatrix} \tag{27}$$

This normal is now set equal to the rotated intial normal (Eq.20), and the resulting three equations are solved for λ and μ, now as a function of a, u, and v. These are the solutions:

$$\begin{aligned}
\sin\lambda &= \frac{\sin a \sin u \cos v - \sin v \cos a}{\sqrt{\cos^2 u \cos^2 v + (\sin a \sin u \cos v - \cos a \sin v)^2}} \\
\cos\lambda &= \frac{\cos u \cos v}{\sqrt{\cos^2 u \cos^2 v + (\sin a \sin u \cos v - \cos a \sin v)^2}} \qquad (28) \\
\dot{\lambda} &= \dot{a}\frac{\cos u \cos v(\cos a \sin u \cos v + \sin a \sin v)}{\cos^2 u \cos^2 v + (\sin a \sin u \cos v - \cos a \sin v)^2} \qquad (29) \\
\ddot{\lambda} &= \frac{\ddot{a}}{\dot{a}}\dot{\lambda} - \dot{\lambda}^2(\sin a \tan u - \cos a \frac{\tan v}{\cos u})\left[1 + \frac{1 - \frac{1}{2}\sin 2a \sin u \sin 2v}{(\cos a \sin u \cos v + \sin a \sin v)^2}\right] \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\sin\mu &= \frac{1}{2}\left[\sqrt{1 + \cos a \sin u \cos v + \sin a \sin v} - \sqrt{1 - \cos a \sin u \cos v - \sin a \sin v}\right] \\
\cos\mu &= \frac{1}{2}\left[\sqrt{1 - \cos a \sin u \cos v - \sin a \sin v} - \sqrt{1 + \cos a \sin u \cos v + \sin a \sin v}\right] \qquad (31) \\
\dot{\mu} &= -\frac{\dot{a}(\sin a \sin u \cos v - \cos a \sin v)}{2\sqrt{1 - (\cos a \sin u \cos v + \sin a \sin v)^2}} \qquad (32) \\
\ddot{\mu} &= \frac{\ddot{a}}{\dot{a}}\dot{\mu} + \dot{\mu}\dot{a}\left(\frac{\cos a \sin u \cos v + \sin a \sin v}{\sin a \sin u \cos v - \cos a \sin v}\right)\left[1 - 4\left(\frac{\dot{\mu}}{\dot{a}}\right)^2\right] \qquad (33)
\end{aligned}$$

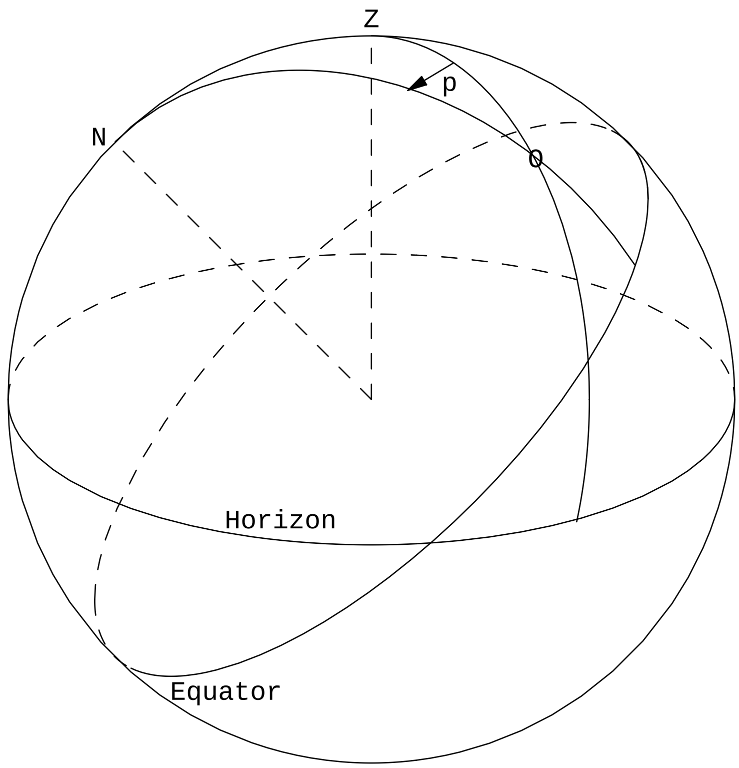
Equations 28-33 reduce to equations 21-26 when $v = 0$.

6 Additional Complications

As mentioned in the introduction, the equations of motion are not the only complications one has to worry about when allowing for “off-altitude-axis” positions on the Nasmyth platforms. The amount of field rotation is altered, and because the incoming lightbeam will see different projections of the actual shape of the tertiary mirror some vignetting may occur.

6.1 Field Rotation

It is important to note the meaning of field rotation in this paper. Perhaps in contrast to the conventional use of the word, ’field rotation’ here is not used to denote the *rate* of rotation of an image on a detector, but the *degree* of rotation. By definition the field rotation at the prime or Cassegrain focus, i.e. on the optical axis, is equal to zero when the primary is pointing at the meridian. When the primary points at an object away from the meridian the field rotation is given by the angle between the great circle perpendicular to the horizon and passing through the object to the great circle perpendicular to the equator and passing through the object. This angle is more commonly known as the parallactic angle (p).

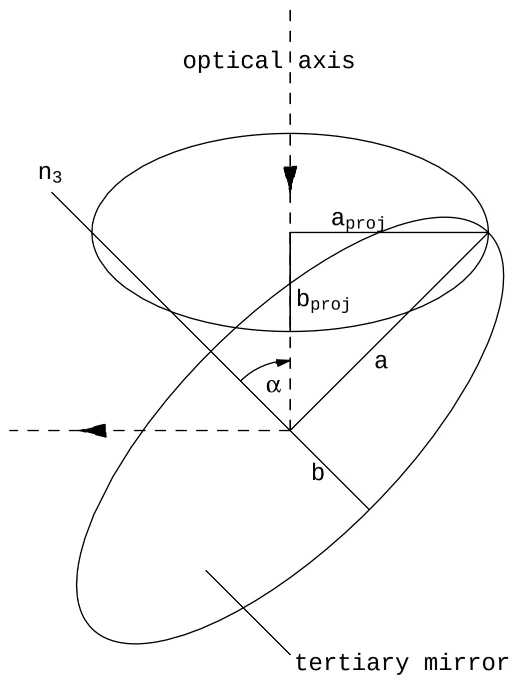


Nelson (1981) has worked out the dependence of this field rotation and its first and second time derivatives on the latitude of the telescope and the hour angle and declination of the observed object. The results are shown in eq. 3, 6, and 9. The field rotation at the Nasmyth platforms is slightly different, and given by:

$$p_N(u) = p + a \cos u \cos v \tag{34}$$

6.2 Vignetting

The tertiary will be tilted at an angle $(45^\circ - \mu)$ with respect to the optical axis in order to reflect the incoming light into the detector. The incoming light will "see" the projection of the surface of the tertiary mirror onto the optical plane, i.e. the plane perpendicular to the optical axis. For the traditional Nasmyth platform position ($u = 0, \mu = 0$) this projection has the shape of a circle when the tertiary mirror's shape is an ellipse with eccentricity $e = \frac{1}{\sqrt{2}}$. For "off-altitude-axis" positions on the Nasmyth platform the angle between the optical axis (z')and the tertiary normal (n_3) and thus the projected shape of the tertiary mirror will vary with time. For certain configurations it is possible that the incoming light will "see" an ellipse of smaller area than the traditional circle, in which case some of the light would not be reflected into the detector, and the image would experience some vignetting.



Let a denote the semi-major axis and b the semi-minor axis of the tertiary mirror ellipse. Let α denote the angle between the optical axis (z') and the tertiary normal (n_3): $n_3 \cdot \hat{z}' = \cos\alpha = \cos(\frac{\pi}{4} - \mu)$. An eccentricity of $\frac{1}{\sqrt{2}}$ implies that $\frac{a}{b} = \sqrt{2}$. The projection of the tertiary mirror ellipse onto the optical plane has no effect on the length of the semi-minor axis, but changes the length of the semi-major axis.

$$\frac{a_{projected}}{b_{projected}} = \frac{a}{\sqrt{2}b}\cos(\frac{\pi}{4} - \mu) = \cos\mu + \sin\mu \tag{35}$$

Using eq.24 this gives the following result for the ratio of projected semi-major axis ($a_{projected}$) to semi-minor axis (b).

$$\frac{a_{projected}}{b} = \sqrt{1 + \cos a \sin u \cos v + \sin a \sin v} \tag{36}$$

Note that this result shows that vignetting occurs only for $(\cos a \sin u \cos v + \sin a \sin v) < 0$, i.e. only if the detector is positioned "behind" the primary mirror. How far behind the primary the detectors can lie before the amount of vignetting becomes intolerable is dependent on the actual design of the telescope (e.g. the distance of the tertiary above the primary, etc.) and will not be adressed here.

7 Summary

Allowing for "off-altitude-axis" detector positions on the Nasmyth platforms greatly increases the flexibility of a telescope. The price to pay is that the tertiary mirror cannot remain fixed with respect to the primary as the telescope tracks an object. In this paper I have derived the equations of motion of the tertiary mirror (Eq.21-26 for $v = 0$ and Eq. 28-33 for $v \neq 0$) and calculated the additional field rotation (Eq.27) as well as the extent of vignetting (Eq.29). The motion of the tertiary mirror, necessary in order for the incoming light to be reflected into a detector located at a position away from the altitude axis, is not subject to any blind spots. The additional field rotation does not cause any complications and vignetting occurs only when the detector is positioned "behind" the primary mirror ($(\cos a \sin u \cos v + \sin a \sin v) < 0$).